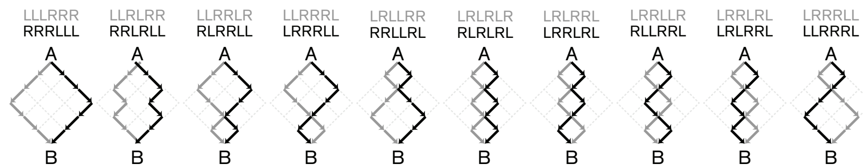
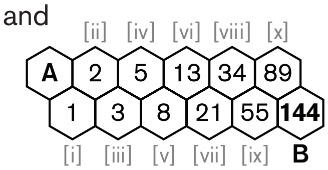
MATHCOUNTS® Minis December 2018 Activity Solutions

Warm-Up!

1. Any path from A to B that requires only moves down to the right and down to the left will include exactly three moves down to the left (L) and exactly three moves down to the right (R) in some order. We can calculate the number of way to arrange the letters LLLRRR. There is a formula for counting permutations of objects where some of the objects are identical, or undistinguishable. We can use the formula $r!/(n_1! \times n_2! \times \cdots)$ to determine the number of arrangements of r objects with n_1 identical bjects of one kind, n_2 identical objects of another, and so forth. In this case, we have 6 letters, of which, 3 are Ls and 3 are Rs. As the figure shows, there are a total of $6!/(3! \times 3!) = (6 \times 5 \times 4)/(3 \times 2 \times 1) = 5 \times 4 =$ **20** different paths, as described.

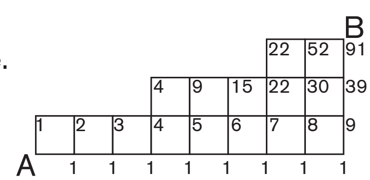


2. We are asked to count the paths from cell A to cell B. For convenience, we have labeled the 10 cells between them with the roman numerals [i] through [x]. Starting from A, there is only 1 way to get to [i], and that is from A. Again, starting from A, there are 2 ways to get to [ii]: from A, and from [i]. Next we get to [iii] from either [i] or [ii], and since there is 1 way to get to [i] and 2 ways to get [ii], there are 1 + 2 = 3 ways to get to [iii]. If we continue in this manner, to count the number of ways to get to each of [iv] through [x], we find that there are **144** paths from A to B, as this figure shows.



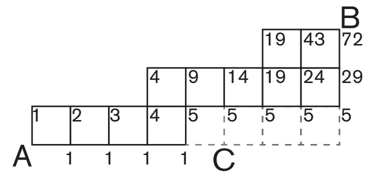
3. Since there is only one M, any path spelling MATH must begin there. Then there are 4 ways to spell MA. Notice that his diagram is symmetrical in such a way that the letters surrounding each A are the same. That means the number of desired paths through each A will be the same. So, let's just look at the A to the right of M. From this A, there are 3 ways to spell MAT. Now, notice that the letters surrounding two of the Ts are the same, but those surrounding the third T are different. From the T above A and from the T below A, there are 2 ways to spell MATH for each. From the T to the right of A there are 3 ways to get MATH. Thus, there are 2 + 2 + 3 paths. But remember this is true for paths through each of the 4 As. The total number of MATH paths is (2 + 2 + 3) × 4 = 7 × 4 = **28**.

4. Moving from A to B, we can only go up or right one segment at a time. If we look at the intersection one segment up from A, we can see there is only 1 way to get to this location. Similarly, there is only 1 way to get the the intersection one segment to the right of A. However, if we look at the intersection diagonal from A, we see that we can get to it in 2 ways, we can go up one and right one or we can go right one and up one. As we go through to each next intersection, we will observe that the number of ways to arrive at that location is the sum of the number of ways to arrive at two intersections to the left or below (note: this method will be explained in detail in the video). Following this pattern we can continue to sum the paths to each intersection until we arrive at our destination, B, and have calculated that there are **91** paths to that location.



The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems



5. This problem uses the same figure as problem 4, but we now have the added point C to avoid. Since we cannot go through the point C, we can eliminate any segments that go connect to C and any segments that can only be reached by passing through C. The adjusted figure is shown with those segments grayed out. Using the same counting method from problem 4 and the video. We can see that with the addition of point C, there are now only **72** paths from A to B that do not pass through C.

6. Moving from the top of the triangle downward to spell MATHEMATICS, we start at M and have 2 options for A. Then from each A there are another 2 options. This means there are 2 × 2 = 4 ways to spell MAT. From each T, there are again 2 options to move to H. There are 2 × 2 × 2 = 8 ways to spell MATH. This pattern continues until we get all the way to S making the total number of ways to spell out MATHEMATICS 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 = 2^{10} = **1024** ways.

Alternatively, if you wanted to count the number of paths to each letter as you move down the triangle, you will get the pattern shown here. This is Pascal's Triangle written out to row 10. The final answer can be found by summing all of the numbers across row 10. The sum you will find is 1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1 = **1024** ways.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

7. If we start at one of the 4 N's in a corner, then we have only 1 choice for the first O, then 3 choices for the second O, then 5 choices for the second N, for a total of 4 × 1 × 3 × 5 = 60 ways to form NOON, starting from a corner. If we start at one of the 8 N's on the side, then we have 2 choices for the first O, then 3 choices for the second O. If we choose the second O adjacent to our original N, then we only have 4 choices for the final N, otherwise (for the two choices of the second O) we have 5 choices for the final N. Thus we have 8 × 2 × (2 × 5 + 1 × 4) = 224 ways to form NOON, starting from a side. This gives a total of 60 + 224 = **284** ways to form NOON.