MATHCOUNTS®

Order of Operations & Defining New Rules

Warm-Up!

Try these problems before watching the lesson.

***Coach instructions:** These problems are taken from old sprint and CDRs. They should be solvable in 45 to 80 seconds on average, but for this exercise, give students 12 minutes to go through the problems (2 minutes per problem). They should be done without the use of a calculator.*

***Note:** The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

1. What is the value of $4 \times (50 + 7)$?

   Following the order of operations, we solve $4 \times (50 + 7) = 4 \times 57 = 228$. If your students are familiar with distrubuting terms, you can point out that it might be an easier computation: $4 \times 50 + 4 \times 7 = 200 + 28 = 228$.

2. What ***common fraction*** is equivalent to $1\frac{1}{2} + \frac{6}{5} - 0.25$?

   Converting all terms to fractions, we get 3/2 + 6/5 – 1/4. Next, let's find a common denominator and solve. We can rewrite the expression as 30/20 + 24/20 – 5/20 = 49/20.

3. What is $0 \cdot 1 + \frac{0}{1} + 0^1 + 1^0$?

   Following the order of operations, we solve $0 \cdot 1 + 0/1 + 0^1 + 1^0 = 0 \cdot 1 + 0/1 + 0 + 1 = 0 + 0 + 0 + 1 = 1$.

4. What is the value of $(10 - 5)^2 + 12 \div 4$?

   Following the order of operations, we solve $(10 - 5)^2 + 12 \div 4 = 5^2 + 12 \div 4 = 25 + 12 \div 4 = 25 + 3 = 28$.

5. What is the value of $9(\frac{1}{3} + 2 - \frac{2}{3})$?

   Following the order of opertations, we solve 9(1/3 + 2 – 2/3) = 9(7/3 – 2/3) = 9(5/3) = 15. Similar to problem 1, this is another opportunity to use distribution of terms: 9(1/3 + 2 – 2/3) = 9 × 1/3 + 9 × 2 – 9 × 2/3 = 3 + 18 – 6 = 15.

6. What is the value of $100 - \frac{10}{0.1}$?

   Following the order of operations, we solve 100 – 10/0.1 = 100 –100 = 0.

Coach instructions: *After students take time to attempt the warm-up problems, play the video and have them follow along in the lesson and solutions. The video run time is ~10 mintues.*

The Problems

Take a look at the following problems and follow along as they are explained in the video.

7. Define the operation $a \# b = a^2 + b$. What is the value of $(2 \# 1) \# (2 \# 1)$?

   Solution in video. Answer: 30.

8. If $a \bigstar b = a + b - 1$, what is the value of $5 \bigstar 5 \bigstar 5 \bigstar 5 \bigstar 5$?

   Solution in video. Answer: 21.

9. If $a \blacklozenge b$ is defined as $a \cdot b + 3$, what is the ***absolute difference*** between $(10 \blacklozenge 11) \blacklozenge 12$ and $10 \blacklozenge (11 \blacklozenge 12)$?

   Solution in video. Answer: 6.

Coach instructions: *Have students try problems that will piece together their knowledge of order of operations and some of the strategies from the video. These are intended to be done without a calculator. The average difficulty has increased from the warm-up. Give students 12-16 minutes to work on these (3-4 minutes per problem).*

Piece It Together

Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

10. What is the value of $(x + \frac{1}{x})^2$, if $x = \sqrt{\frac{5}{8}}$? Express your answer as a common fraction.

    Substituting in $\sqrt{5/8}$ for x, we get $(\sqrt{5/8} + 1/\sqrt{5/8})^2 = ((5/8)/(\sqrt{5/8}) + 1/\sqrt{5/8})^2 = ((13/8)/(\sqrt{5/8}))^2 = (169/64)/(5/8) = 169/40$. Alternatively, if your students know how to expand $(x + 1/x)^2$ to get $x^2 + 2 + 1/x^2$, this might be a simpler calculation for them. Here they can plug in 5/8 for the x^2 terms and solve $5/8 + 2 + 1/(5/8) = 5/8 + 2 + 8/5 = 25/40 + 80/40 + 64/40 = 169/40$.

11. If $x \triangle y = x + y - |x - y|$, what is the value of $(3 \triangle 4) - (2 \triangle 1)$?

    Solving $(3 \triangle 4) = 3 + 4 - |3 - 4| = 7 - |-1| = 7 - 1 = 6$ and then $(2 \triangle 1) = 2 + 1 - |2 - 1| = 3 - |1| = 3 - 1 = 2$, we can determine $(3 \triangle 4) - (2 \triangle 1) = 6 - 2 = 4$.

12. If $a \,⌘\, b = \frac{ab}{a+b}$ and $a \,⌘\, 4 = 3$, what is the value of a?

    This will require the use of algebra to solve for the unknown. We can set up the equation $4a/(a + 4) = 3$. Multiplying both sides by $a + 4$, we get $4a = 3a + 12$. Finally, subtracting $3a$ from both sides gives the final answer of $a = 12$.

13. Joanna forms an arithmetic expression using each of $\frac{1}{10}$, $3\frac{1}{2}$ and $2\frac{4}{5}$ exactly once and using each of the two operators + and ÷ exactly once with as many sets of parentheses as she wishes. What is the ***absolute difference*** between the greatest and least possible values of Joanna's expression? Express your answer as a ***mixed number***.

This requires some understanding of how order of operations can affect the final value and also how division of fractions affects the final value. Let's start by considering the largest number we can create. Dividing by a number greater than one will result in a smaller value, where as dividing by a number less than one will result in a larger value. This means we need to divide by 1/10 — the smallest available value. This leaves us with the addition between 3 1/2 and 2 4/5. The last decision is the use of the parentheses. We want to divide the largest possible value by 1/10 so we should put the parentheses around our addition giving us (3 1/2 + 2 4/5) ÷ 1/10 = 6 3/10 ÷ 1/10 = 63. Next we need to consider the smallest possible value. This will be achieved by dividing by the largest value. If we take the mirror image of our previous expression, we get 1/10 ÷ (3 1/2 + 2 4/5) = 1/10 ÷ 6 3/10 = 1/10 ÷ 63/10 = 1/63. So the absolute difference is 63 − 1/63 = 62 62/63.

Optional Extension

To extend your understanding and have a little fun with math, try the following activities.

Create a rule for *a* ☺ *b* that always equals 1 no matter what two numbers are used for *a* and *b*. Get creative! Make more than one! See which of your friends came up with the most complex but successful rule!

There are infinite answers to this. It might be beneficial to implement some rules or guidelines for your students — you must use a certain number of operations, you have to use specific operations a specific number of times, you can or can't use parentheses, etc. As currently written, here are a few rule examples: a ☺ b = 1 + a − a + b − b or a ☺ b = a + b ÷ b − a or a ☺ b = $(a - 1)^2 - a^2 - 2a$. If your mathletes have difficulty getting started, share one of these examples!

Come up with a rule that is challenging to solve. Switch with your friends and see if you can stump them! Note: agree with your friends on a maximum number of steps or operators.

Similar to the previous example, it might be beneficial to introduce some guidlines!

Where are these questions from?	
1. From 2014 Chapter CDR, #35	*7. From 2009 National CDR, #63*
2. From 2014 Chapter CDR, #22	*8. From 2013 National CDR, #53*
3. From 2014 Chapter CDR, #16	*9. From 2016 National CDR, #18*
4. From 2014 Chapter CDR, #80	*10. From 2016 State CDR, #20*
5. From 2014 State CDR, #43	*11. From 2016 Chapter CDR, #52*
6. From 2017 Chapter Sprint, #9	*12. From 2014 Chapter Sprint, #9*
	13. From 2016 National Sprint, #4