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DSEC DEFENSE STEM EDUCATION CONSORTIUM
MATHCOUNTS®

*This practice plan was created by **Taren Long**, a math teacher and coach at Chesapeake Public Charter School. Taren created numerous free resources for MATHCOUNTS coaches in her role as the 2020-2021 DoD STEM Ambassador for MATHCOUNTS. Find more resources and information at **dodstem.us**.*

Averages

Warm-Up!

Try these problems before watching the lesson.

***Coach instructions:** Give students around 10 minutes (2 minutes per problem) to go through the warm-up problems.*

***Note:** The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

1. In the set 2, 5, 11, 17, 20, what is the difference between the ***mean*** and the ***median***?

   The mean of the set is $\frac{2+5+11+17+20}{5}=\frac{55}{5}=11$. The median of the set is also 11. 11 − 11 = 0.

2. What is the ***average*** of the integers from 13 to 31, inclusive?

   When you add the numbers from 13 to 31, notice that each pair (for example, 13 + 31) total 44 and average to 22. You could also add the numbers, and divide by the 19 numbers $\frac{13+14+15...+29+30+31}{19}=22$.

3. If the ***average*** of 20 numbers is 16, what is their sum?

   The average of 20 numbers can be written as $\frac{x}{20}=16$, where *x* is the sum of the 20 numbers. Isolating the variable, $x=320$.

4. What is the ***mean*** of 3*x*, 4*x* – 5 and 2*x* – 1?

   The mean of the set can be written and simplified: $\frac{3x+(4x-5)+(2x-1)}{3}=\frac{3x+4x+2x-5-1}{3}=\frac{9x-6}{3}=3x-2$.

5. What is the ***mean*** of all three-digit numbers that can be created using each of the digits 1, 2 and 3 exactly once?

   Listing out three-digit numbers using 1, 2 and 3 exactly once gives: 123, 132, 213, 231, 321 and 312. The mean of the numbers is $\frac{123+132+213+231+321+312}{6}=\frac{1332}{6}=222$.

The Problems

Coach instructions: *After students try the warm-up problems, play the video and have them follow along with the solutions.*

Take a look at the following problems and follow along as they are explained in the video.

6. What is the ***mean*** of seven numbers if the ***mean*** of the first two is 11 and the ***mean*** of the last five is 18?

   Solution in video. Answer: 16.

7. Hadley scored 92, 73, 79 and 87 points on the first four tests of the quarter. There is one test remaining. What is the minimum number of points that Hadley must score on the final test in order to have a ***mean*** of 80 points for the five tests?

   Solution in video. Answer: 69.

8. The ***arithmetic mean*** of four numbers is 15. Two of the numbers are 10 and 18 and the other two are equal. What is the product of the two equal numbers?

   Solution in video. Answer: 256.

Piece It Together

Coach instructions: *After watching the video, give students 10 minutes to try the next four problems.*

Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.

9. Louis received grades of 75, 90 and 81 on his unit tests. The final exam is worth three times as much as a unit test. What grade must Louis make on the final exam to end up with an ***average*** of 86 on all his tests?

   The unit tests together are valued equally with the final exam. Their sum is 246, and their average is $^{246}/_{3}$ = 82. 82 is four units below the goal of 86, so Louis must score four points higher than 86 on the final exam to earn the 86 average. 86 + 4 = 90.

10. The ***mean*** of four distinct positive integers is 5. If the largest of these four integers is 13, what is the smallest of the four integers?

    The four distinct positive integers must have a sum of 20. If the largest integer is 13, the other three digits must sum to 20 – 13 = 7. The only three distinct positive integers that add to seven are 4, 2 and 1, the smallest of which is 1.

11. From 11 positive integer scores on a 10-point quiz, the ***mean*** is 8, the ***median*** is 8, and the ***mode*** is 7. Find the maximum number of perfect scores possible on this test.

With 11 integers, the middle integer must be equal to 8. This leaves a maximum of 5 scores larger than 8 that could be equal to 10 points, but since 7 is the mode, there can be only 4 perfect scores remaining. This would require the set 7, 7, 7, 7, 7, 8, __, 10, 10, 10, 10. Using the lowest possible number for the __, 8, the mean is too high, which means four perfect scores is not possible. Reducing it to three 10s gives at least one possible solution of 6, 7, 7, 7, 7, 8, 8, 8, 10, 10, 10. Thus, the answer is 3.

12. The ***mean*** of six positive integers is 5 and the ***median*** is 6. What is the largest the ***mode*** could be, given that the ***mode*** is unique?

Among six integers with a median of 6, setting four digits as 6 sums to 24, which leaves the last two digits as 3 to sum to 30. If we want to try for a mode of 7, we can only have three digits be 7, which leaves _, _, 5, 7, 7, 7, which have a sum of 26. So, the last two digits can be 2, 2 or 1, 3. To try for a mode of 8, we need to reduce the lower numbers. _, _, 4, 8, 8, 8 has a sum of 28. So, our last two digits can be 1 and satisfy all of the conditions. If we try to make 9 the mode, _, _, 3, 9, 9, 9 already adds to 30, so we would have to reduce the instances of the 9 to two. The median 6 means the middle terms must sum to 12, but 12 + 18 = 30, so _, _, 5, 7, 9, 9 is impossible. Thus, the largest mode possible is 8.

Optional Extension

Coach instructions: *Once your students have completed the problems and feel they have a comfortable understanding of the concept, let them try this open-ended task.*

To extend your understanding and have a little fun with math, try the following activities.

There are quick ways to add sequences of numbers, which makes finding the mean of a sequence of numbers much easier. Try to come up with a formula or strategy to figure out the average of any consecutive sequence of numbers.

Consider:

- Sequences with an odd or even number of terms.
- Sequences that begin with 1 and sequences that do not begin with 1.
- Sequences of only odd numbers or only even numbers (or otherwise non-consecutive sequences).

To average any large sequence of consecutive numbers, you can use the formula $\frac{x_1+x_n}{2}$ This formula works whether sequences have an odd or even number of terms, as well as with sequences with different starting points.

Interestingly, this formula also works for the average of sequences with repetitive skips, such as only odd numbers or skipping additional numbers, such as in the linear sequence 7, 12, 17, 22, 27, 32.